Momentum-Impulse Study Guide
Elliot Silva with Technical Aid from Mr. Papaleo

Background: In the study of classical mechanics, a fundamental concept is the study of momentum, impulse, and their relationship. This unit discusses the relationship between forces, momentum, impulse, and consequently, elastic, inelastic, and completely inelastic collisions. As a real life application of this, consider the tennis racket below, which delivers an impulse to the tennis ball, thereby changing its velocity.

$$\sum \vec{F} = m\vec{a}$$
$$= m\frac{d\vec{v}}{dt}$$
$$= \frac{d}{dt}\left(m\vec{v}\right)$$

Therefore, force is the time rate of change of the product of mass and velocity.
The product of mass and velocity is called linear momentum, "p", units: kgm/s
$\vec{p} = m\vec{v}$, where momentum is a vector quantity in the same direction as velocity.

So, $\sum \vec{F} = \frac{d\vec{p}}{dt}$

A force, F, acts on a particle for a given period of time, from t_1 to t_2
Impulse is defined as the summation of all the forces over a given time.

$$\vec{J} = \sum \vec{F}dt$$
$$= \left(\frac{d\vec{p}}{dt}\right)dt$$
$$= d\vec{p}$$

So, we also see that impulse is the change in momentum.

$$\vec{J} = \vec{p}_2 - \vec{p}_1$$
$$\vec{p}_2 = \vec{J} + \vec{p}_1$$

Substituting for momentum with mass times velocity,

$$\vec{J} = m\vec{v}_2 - m\vec{v}_1$$
$$m\vec{v}_2 = \vec{J} + m\vec{v}_1$$
$$\vec{v}_2 = \vec{J}/m + \vec{v}_1$$

This assumes that the mass is constant before and after the collision. If not, we will have to use conservation of momentum, which we'll get to in just a moment.

Graphically speaking, impulse is really the area under the curve of a force vs. time graph from t_1 to t_2. However, very often instead of being given the equation for force and having to use an integral from t_1 to t_2 to find the impulse, we will most simply be given the average force acting on the object over a given time interval. Therefore, because the force is a constant, we do:

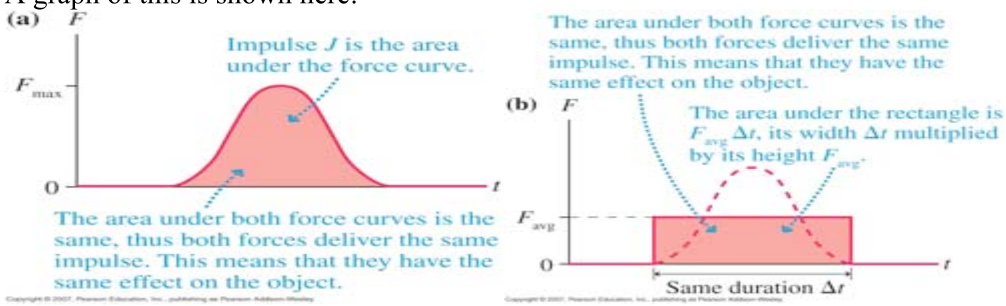
$$\vec{J} = \int_{t_1}^{t_2} F_{ave}\,dt$$

$$= F_{ave}\int_{t_1}^{t_2} dt$$

$$= F_{ave}(t_2 - t_1)$$

$$= F_{ave}\Delta t$$

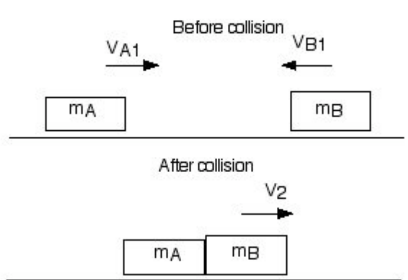
A graph of this is shown here:



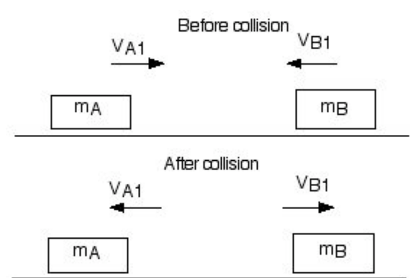
Conservation of Momentum: When any two bodies in a closed system interact, as long as there is no net external force, then momentum is conserved. This means that $\vec{P}$, the total momentum of a system is the same before and after any interaction (aka "collision") between two or more bodies in a closed system. [$\vec{P}_1 = \vec{P}_2$] Note that this relationship is true for momentum in every direction, meaning $\vec{P}_{x1} = \vec{P}_{x2}$, and $\vec{P}_{y1} = \vec{P}_{y2}$.

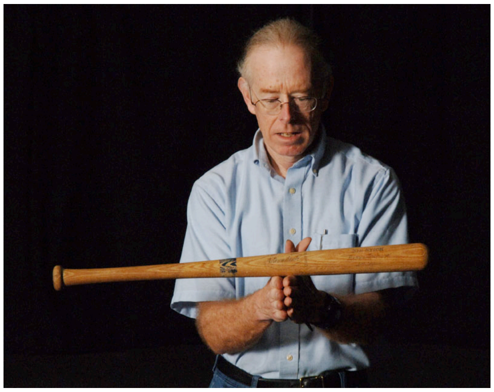
Three types of collisions:
ELASTIC: 2 objects collide and bounce off each other, kinetic energy conserved
INELASTIC: 2 objects collide but do not stick together, kinetic energy not conserved
COMPLETELY INELASTIC: 2 objects collide and stick together, moving as one

Elastic Collision:



Inelastic Collision:





Center of Mass: This is the point where we can consider all the mass of the object to be located, at the center of any uniform sphere, cube, etc. When the object is not uniform, however, the center of mass can no longer be assumed to be in the center. See Joe Biden balancing the baseball bat to your left.

Furthermore, we can also determine the center of mass of a system of point masses. Imagine a beam with weigths of different masses on each end. The balance point will not be in the middle; it will be closer the larger mass. So, to find the coordinates of the center of mass we do:

$\vec{x}_{cm} = \dfrac{m_1\vec{x}_1 + m_2\vec{x}_2 + \ldots + m_n\vec{x}_n}{m_1 + m_2 + \ldots + m_n}$, and $\vec{y}_{cm} = \dfrac{m_1\vec{y}_1 + m_2\vec{y}_2 + \ldots + m_n\vec{y}_n}{m_1 + m_2 + \ldots + m_n}$, n = # of masses.

Next, we can say that the object acts as a single object with mass $M = m_1 + m_2 \ldots m_n$. So,

$\vec{x}_{cm} = \dfrac{m_1\vec{x}_1 + m_2\vec{x}_2 + \ldots + m_n\vec{x}_n}{M}$ => $M\vec{x}_{cm} = m_1\vec{x}_1 + m_2\vec{x}_2 + \ldots + m_n\vec{x}_n$, differentiate with respect to t: $M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$. Therefore, the total linear momentum of a system is the sum of the momenta of all the individual particles in that system. Differentiating again gives: $\vec{F}_{net} = M\vec{a}_{cm}$, proving that if the net external force acting on a system is zero, then there will be no acceleration at the center of mass.

Key Vocab:
Conservation of Linear Momentum: the momentum before a collision is the same as the momentum after the collion.
Elastic collisions: conserve kinetic energy
Inelastic collisions: don't conserve kinetic energy
Competely elastic collisions: objects move as one, don't conserve kinetic energy
Impulse: the product of the force and the time over which that force acts
Impulse/Momentum Theory: impulse on an object is equal to change in momentum.

Key Formulae:

$\vec{p} = m\vec{v}$ $\quad \vec{P}_1 = \vec{P}_2$ $\quad \vec{J} = \int_{t_1}^{t_2} F_{ave}\,dt$ $\quad \vec{J} = F_{ave}\Delta t$

$\vec{x}_{cm} = \dfrac{m_1\vec{x}_1 + m_2\vec{x}_2 + \ldots + m_n\vec{x}_n}{M}$ $\quad M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$

PROBLEM #1: Object 1 of mass m moves along the x-axis with velocity $\vec{v}$ toward Object 2 of mass 2m. After impact Object 1 moves at $\vec{v}'_1$ at 30° and Object 2 moves at $\vec{v}'_2$ at 45°. (a) Find $\vec{v}'_1$ and $\vec{v}'_2$ in terms of $\vec{v}$ after the collision. (b) Determine if the collision is elastic.

(a) Conservation of momentum tells us:
x-component: $m\vec{v} = m\vec{v}'_1\cos 30° + 2m\vec{v}'_2\cos 45°$
y-component: $0 = m\vec{v}'_1\sin 30° - 2m\vec{v}'_2\sin 45°$
Add these to get: $m\vec{v} = m\vec{v}'_1(\cos 30° + \sin 30°)$

And we can solve for $\vec{v}'_1$: $\vec{v}'_1 = \dfrac{\vec{v}}{\cos 30° + \sin 30°} = \dfrac{2\vec{v}}{1+\sqrt{3}}$

Plugging $\vec{v}'_1$ into the first equation gives: $m\vec{v} = \mathrm{m}\dfrac{2\vec{v}}{1+\sqrt{3}}\cos 30° + 2\mathrm{m}\,\vec{v}'_2\cos 45°$ =>

$$2m\vec{v}'_2 \sin 45° = m\frac{2\vec{v}}{1+\sqrt{3}}\sin 30° \Rightarrow \vec{v}'_2 = \frac{\frac{2\vec{v}}{1+\sqrt{3}}\sin 30°}{2\sin 45°} \Rightarrow \vec{v}'_2 = \frac{\vec{v}}{\sqrt{2}(1+\sqrt{3})}$$

(b) The collision is elastic only if kinetic energy is conserved, so if K'=K:

Kinetic energy after collision: $K' = \frac{1}{2}m\vec{v}_1^2 + \frac{1}{2}2m\vec{v}_2^2$ =>

$$K' = \frac{1}{2}m\left(\frac{2\vec{v}}{1+\sqrt{3}}\right)^2 + \frac{1}{2}2m\left(\frac{\vec{v}}{\sqrt{2}(1+\sqrt{3})}\right)^2 \Rightarrow K' = \frac{m\vec{v}^2}{(1+\sqrt{3})^2}$$

K'=K if the coefficients of $m\vec{v}^2$ are equal. However, $\dfrac{1}{(1+\sqrt{3})^2} < 2$, therefore the collision is inelastic.

PROBLEM #2: Two objects, one of mass m and one of mass 2m, hang from light threads from the ends of a uniform bar of length 3L and mass 3m. Mass m hangs at a distance L and mass 2m hangs at a distance 2L. Find the center of mass.

Answer: Each object can be treated as a point mass, with the center of mass of the uniform bar at the origin. So we have 3m @ (0,0), m @ (-3L/2,-L), and 2m @ (3L/2,-2L). Using the formula for the center of mass at the x- and y- coordinates separately:

$$\bar{x}_{cm} = \frac{m_1\bar{x}_1 + m_2\bar{x}_2 + ... + m_n\bar{x}_n}{M} = \frac{(m)(-3L/2) + (2m)(3L/2) + (3m)(0)}{m + 2m + 3m} = \frac{3mL/2}{6m} = \frac{L}{4}$$

$$\bar{y}_{cm} = \frac{m_1\bar{y}_1 + m_2\bar{y}_2 + ... + m_n\bar{y}_n}{M} = \frac{(m)(-L) + (2m)(-2L) + (3m)(0)}{m + 2m + 3m} = \frac{-5mL}{6m} = -\frac{5L}{6}$$

Therfore the center of mass is at ($\bar{x}_{cm}, \bar{y}_{cm}$) = (L/4,-5L/6)

PROBLEM #3: A sphere m_1=1000kg initially moving at $\mathbf{v}_1$=12m/s collides with a sphere of unknown mass, m_2. The collision is completely inelastic and after the collision the two spheres move at velocity $\vec{v}(t) = \dfrac{8}{1+5t}$ in m/s. (a) Calculate m_2. (b) Calculate the impulse from t=0 to t=2s.

(a) Conservation of momentum gives $m_1\vec{v}_1 = (m_1 + m_2)\vec{v}(0) \Rightarrow m_2 = \dfrac{m_1\vec{v}_1}{\vec{v}(0)} - m_1 =$ (1000kg)*(12m/s)/[8/(1+5(0))]-1000kg = 500kg

(b) Using the Impulse/Momentum Theory, $\vec{J} = \Delta m\vec{v} = (m_1 + m_2)\vec{v}(2) - (m_1 + m_2)\vec{v}(0) =$ (1000kg+500kg)*[8/(1+5(2s))]-(1000kg+500kg)*[8/(1+5(0))] = -10909.09Ns, where the negative sign just means that the force is acting against the motion of the system.